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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 77, 425–432 (1980)

Non-Linear Multiparameter Eigenvalue Problems for Ordinary Differential Equations

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Submitted by J. Cronin

In this paper we consider the non-linear multiparameter eigenvalue problem in ordinary differential equations: $y_r''(x_r) + q_r(x_r) y_r(x_r) + M_r(y_r) + \sum_{s=1}^{n} \mu_s(a_{rs}(x_r + P_{rs})\, y_r(x_r) = 0$, $r = 1,\ldots, n$, where the non-linear functions M_r and P_{rs} will be required to satisfy certain Fréchet differentiability conditions. We study bifurcation from the simple eigenvalues of the corresponding linear problem and state a completeness theorem for the eigenfunctions of these systems.

1. INTRODUCTION

In recent years interest has been directed at multiparameter eigenvalue problems for ordinary differential equations. Typically, these problems may be formulated via the coupled system of linear ordinary differential equations

$$\frac{d^2 y_r(x_r)}{dx_r^2} + q_r(x_r)\, y_r(x_r) + \sum_{s=1}^{n} \lambda_s a_{rs}(x_r)\, y_r(x_r) = 0, \quad r = 1,\ldots, n, \tag{1.1}$$

where $x_r \in [a_r, b_r]$, $-\infty < a_r < b_r < \infty$, λ_s are complex spectral parameters and q_r and a_{rs} are continuous real-valued functions defined on $[a_r, b_r]$, r, $s = 1,\ldots, n$. Non-trivial solutions of each member of (1.1) are sought satisfying the Sturm–Liouville boundary conditions

$$\begin{aligned} &y_r(a_r) \cos \alpha_r + y_r'(a_r) \sin \alpha_r = 0, \qquad 0 \leqslant \alpha_r < \pi, \\ &y_r(b_r) \cos \beta_r + y_r'(b_r) \sin \beta_r = 0, \qquad 0 < \beta_r \leqslant \pi,\ r = 1,\ldots, n. \end{aligned} \tag{1.2}$$

* Research supported in part by National Research Council of Canada Grant No. A9073.

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Apart from the continuity imposed on the coefficient functions a_{rs}, we also require that they satisfy the definiteness condition

$$|A|(x) = \det\{a_{rs}(x_r)\} > 0 \tag{1.3}$$

for all $x = (x_1, ..., x_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]$.

An n-tuple of (necessarily real) numbers $\lambda = (\lambda_1, ..., \lambda_n)$ and a function $y(x) = y_1(x_1) \cdots y_n(x_n)$ are called an eigenvalue and eigenfunction, respectively, if λ and $y_1, ..., y_n$ satisfy non-trivially the system (1.1), (1.2).

Of the properties known to be possessed by the eigenvalues and eigenfunctions of this multiparameter problem, the following are pertinent to the topic of this paper. First, there is the Klein oscillation theorem associated with (1.1)–(1.3). That is, the eigenvalues form a countably infinite set in $\mathbb{R}^n$ and, given an n-tuple of non-negative integers $(p_1, ..., p_n)$, there is a unique eigenvalue λ for which the corresponding eigenfunction $y_1(x_1) \cdots y_n(x_n)$ is such that y_r has precisely p_r zeros in the interval (a_r, b_r), $r = 1, ..., n$. Second, the eigenfunctions form a complete orthonormal set in $L^2([a, b], |A|)$—the space of all complex-valued measurable functions f defined on $[a, b]$ for which

$$\|f\|^2 = \int_{[a,b]} |f(x)|^2 \, |A|(x) \, dx < \infty.$$

The inner product in this space is naturally defined as

$$(f, g) = \int_{[a,b]} f(x) \overline{g(x)} \, |A|(x) \, dx.$$

Details of these and other results in this area may be found in Atkinson [1, 2], Binding and Browne [3], Browne [4, 5], Faierman [7], Källström and Sleeman [8], and Sleeman [10–15].

In this paper we study certain non-linear multiparameter eigenvalue problems which are perturbations of the linear problem discussed above. In particular we shall establish the existence of small bifurcating solutions and give a completeness theorem. The main tool to be used is the following.

THEOREM 1. (The Implicit Function Theorem). *Let B_1, B_2 be Banach spaces and consider a Fréchet differentiable mapping $F(\lambda, x)$: $R \times B_1 \to B_2$. The Fréchet derivative of F is the operator pair (F'_λ, F'_x): $R \times B_1 \to B_2$.*

If $F(\lambda, x)$ is continuously Fréchet differentiable in (λ, x) in a neighbourhood of the origin and

(i) $F(0, 0) = 0$,

(ii) $F'_x(0, 0)$ *has a bounded inverse from B_2 to B_1,* *then there exists, for small* λ, *a function* $x(\lambda)$: $\mathbb{R} \to B_1$ *such that*

(i) $x(0) = 0$,

(ii) $x'(\lambda)$ *exists and is continuous,*

(iii) $F(\lambda, x(\lambda)) = 0$.

Proof. See Sattinger [9, p. 61].

The plan of this paper is as follows. In Section 2 we introduce some notation and set down the non-linear eigenvalue problem to be studied. Section 3 treats the bifurcation problem and the question of completeness.

To conclude this introduction we mention that the results of this paper parallel some recent results of Browne [6] and Thomas and Zachmann [6].

2. Preliminaries

We introduce the following notation. The linear subspace D_r of $L^2([a, b])$ is defined by

$$D_r = \{f_r \in L^2([a_r, b_r]) | f_r' \text{ is } AC; f_r'' + q_r f_r \in L^2([a_r, b_r]);$$

f_r satisfies the boundary conditions (1.2)}. V_{rs}: $L^2([a, b]) \to L^2([a_r, b_r])$ denotes the Hermitean operator $(V_{rs} f_r)(x_r) = a_{rs}(x_r) f_r(x_r)$. We may define a self-adjoint operator T_r: $D_r \subset L^2([a_r, b_r]) \to L^2([a_r, b_r])$ by $T_r f_r = f_r'' + q_r f_r$, $f_r \in D_r$, and further for each $\lambda \in \mathbb{R}^n$ we construct another self-adjoint operator $W_r(\lambda)$: $D_r \subset L^2([a_r, b_r]) \to L^2([a_r, b_r])$ by means of

$$W_r(\lambda) = T_r + \sum_{s=1}^{n} \lambda_s V_{rs}.$$

We shall fix λ to be an eigenvalue of the linear problem (1.1)–(1.3) and u_r to be a non-trivial solution of the rth member of (1.1), (1.2) corresponding to λ and normalized so that $\|u_r\| = 1$, $r = 1, 2, \ldots, n$. The linear space B_r is now given by

$$B_r = D_r \ominus \{u_r\} = \{f_r \in D_r | (f_r, u_r) = 0\}, \qquad 1 \leqslant r \leqslant n,$$

There are several ways in which B_r can be normed so that it is complete and D_r is a closed subspace. For example we have

(i) the graph norm:

$$|||f_r||| = \|f_r\|^2 + \|W_r(\lambda) f_r\|^2$$

with associated inner product

$$\langle f_r, g_r\rangle = (f_r, g_r) + (W_r(\lambda)f_r, W_r(\lambda)g_r).$$

In this case D_r and B_r are, in fact, Hilbert spaces.

(ii) A supremum/graph norm

$$|||f_r||| = \sup_{x_r\in[a_r,b_r]} |f_r(x_r)| + \| W_r(\lambda)f_r\|.$$

(iii) A supremum/graph norm

$$|||f_r||| = \sup_{x_r\in[a_r,b_r]} |f_r(x_r)| + \sup_{x_r\in[a_r,b_r]} |f'_r(x_r)| + \| W_r(\lambda)f_r\|.$$

In checking that these norms have the stated properties we rely heavily on the fact that $W_r(\lambda)$: $D_r \to L^2([a_r, b_r])$ is a closed operator. The details are easy and are left to the reader.

We shall henceforth assume that B_r is normed in one of the above methods.

Finally we let

$$B = \left(\bigoplus_{r=1}^{n} B_r\right) \oplus \mathbb{R}^n.$$

We now state the non-linear problem to be studied.

PROBLEM.

$$T_r y_r + M_r(y_r) + \sum_{s=1}^{n} \mu_s(V_{rs} + P_{rs})(y_r) = 0, \qquad r = 1,..., n, \tag{2.1}$$

where

(i) M_r, P_{rs}: $D_r \to L^2([a_r, b_r])$ are continuously Fréchet differentiable,

(ii) $M_r(0) = P_{rs}(0) = 0$, $M'_r(0) = P'_{rs}(0) = 0$.

3. The Bifurcation Problem

To establish the existence of small bifurcating solutions to (2.1) we use the Lyapounov–Schmidt method described, for example, in Sattinger [9] and make use of the implicit function theorem stated above. To do this we introduce projection operators P_r and Q_r given by

$$P_r v_r = (v_r, u_r)u_r, \quad Q_r v_r = v_r - P_r v_r, \quad v_r \in L^2([a_r, b_r]), \quad 1 \leqslant r \leqslant n.$$

Recall that $u_1(x_1) \cdots u_n(x_n)$ is a suitably normalized eigenfunction corresponding to λ, our fixed eigenvalue for the linear problem (1.1)–(1.3). It is easy to see that $Q_r v_r \in B_r$ if $v_r \in D_r$, $r = 1,..., n$.

We seek solutions y_r and μ_r of (2.1) of the form

$$y_r = \alpha(u_r + U_r), \qquad \alpha \neq 0,$$

$$\mu_r = \lambda_r + t_r, \qquad r = 1,..., n,$$

where $\alpha \in \mathbb{R}$ is a small parameter. Without loss we shall normalize the solutions y_r (should they exist) so that $(\mu_r + U_r, u_r) = 1$. This requires $U_r \in B_r$; $1 \leqslant r \leqslant n$.

Substituting the above forms for y_r and μ_r into (2.1) we see after some simplification that our problem becomes

$$\begin{aligned} T_r U_r &+ \sum_{s=1}^{n} \lambda_s V_{rs} U_r + \alpha^{-1} M_r(au_r + \alpha U_r) \\ &+ \alpha^{-1} \sum_{s=1}^{n} (\lambda_s + t_s) P_{rs}(au_r + \alpha U_r) \\ &+ \alpha^{-1} \sum_{s=1}^{n} t_s(V_{rs} + P_{rs})(au_r + \alpha U_r) = 0, \qquad 1 \leqslant r \leqslant n. \end{aligned}$$

We operate on both sides of this equation with the projections P_r and Q_r to generate the so-called Lyapounov–Schmidt equations for the problem, viz.,

$$\begin{aligned} \alpha^{-1}&(M_r(au_r + \alpha U_r), u_r) \\ &+ \alpha^{-1} \sum_{s=1}^{n} \lambda_s(P_{rs}(au_r + \alpha U_r), u_r) + \sum_{s=1}^{n} t_s(V_{rs}(u_r + U_r), u_r) \\ &+ \alpha^{-1} \sum_{s=1}^{n} t_s(P_{rs}(au_r + \alpha U_r), u_r) = 0 \end{aligned}$$

and

$$\begin{aligned} T_r U_r &+ \sum_{s=1}^{n} \lambda_s V_{rs} U_r + Q_r[\alpha^{-1} M_r(au_r + \alpha U_r) \\ &+ \alpha^{-1} \sum_{s=1}^{n} (\lambda_s + t_s) P_{rs}(au_r + \alpha U_r) \\ &+ \sum_{s=1}^{n} t_s V_{rs}(u_r + U_r)] = 0, \qquad 1 \leqslant r \leqslant n. \end{aligned}$$

We now consider the Hilbert space

$$X = \left[\bigoplus_{r=1}^{n} (L^2([a_r, b_r]) \ominus \{u_r\}) \right] \oplus \mathbb{R}^n$$

and define a mapping F: $\mathbb{R} \times B \to X$ as follows. For

$$(\alpha, U, t) = (\alpha, U_1, ..., U_n, t_1, ..., t_n) \in \mathbb{R} \times B,$$

$F(\alpha, U, t)$ will be the pair of points

$$\left\{ T_r U_r + \sum_{s=1}^{n} \lambda_s V_{rs} U_r + Q_r \left[\sum_{s=1}^{n} t_s V_{rs}(u_r + U_r) + \alpha^{-1} M_r(\alpha u_r + \alpha U_r) \right.\right.$$
$$\left.\left. + \alpha^{-1} \sum_{s=1}^{n} (\lambda_s + t_s) P_{rs}(\alpha u_r + \alpha U_r) \right] \right\}_{r=1}^{n} \in \bigoplus_{r=1}^{n} (L^2([a_r, b_r]) \ominus \{u_r\})$$

and

$$\left\{ \sum_{s=1}^{n} t_s(V_{rs}(u_r + U_r), u_r) + \alpha^{-1}(M_r(\alpha u_r + \alpha U_r), u_r) \right.$$
$$\left. + \alpha^{-1} \sum_{s=1}^{n} (\lambda_s + t_s)(P_{rs}(\alpha u_r + \alpha U_r), u_r) \right\}_{r=1}^{n} \in \mathbb{R}^n.$$

These formulas apply for $\alpha \neq 0$; if $\alpha = 0$, we merely omit the terms in α. The mapping F is continuously Fréchet differentiable in its variables. We wish to solve $F = 0$.

It is easy to see that $F(0, 0, 0) = 0$ and further calculations reveal the Fréchet derivative of F with respect to (U, t) at $(0, 0, 0)$ is the linear mapping from B to X under which the image of (U, t) is the pair of points

$$\left\{ T_r U_r + \sum_{s=1}^{n} \lambda_s V_{rs} U_r + Q_r \sum_{s=1}^{n} t_s V_{rs} u_r \right\}_{r=1}^{n} \in \oplus (L^2([a_r, b_r]) \ominus \{u_r\})$$

and

$$\left\{ \sum_{s=1}^{n} t_s(V_{rs} u_r, u_r) \right\}_{r=1}^{n} \in \mathbb{R}^n.$$

Now given a point $(g_1, ..., g_n, \beta_1, ..., \beta_n) \in X$ we first note that because of the definiteness condition (1.3)

$$\sum_{s=1}^{n} t_s(V_{rs} u_r, u_r) = \beta_r, \qquad r = 1, ..., n,$$

can be solved uniquely for $t_1, \ldots, t_n$ and in a manner which depends continuously on $\beta_1, \ldots, \beta_n$. Note further that since 0 is a simple eigenvalue of $W_r(\lambda)$ with corresponding eigenvalue u_r the map $W_r(\lambda)$: $B_r \to L^2([a_r, b_r]) \ominus \{u_r\}$ is continuous and bijective and so has a continuous inverse by Banach's theorem. Thus having determined $t_1, \ldots, t_n$ above we may now solve uniquely

$$T_r U_r + \sum_{s=1}^{n} \lambda_s V_{rs} U_r = g_r - Q_r \sum_{s=1}^{n} t_s V_{rs} u_r, \qquad r = 1, \ldots, n,$$

for $U_r \in B_r$ and in a manner depending continuously on g_r.

The upshot of these remarks is that the implicit function theorem may be invoked to assert

THEOREM 2. *Under assumption* (1.3), *problem* (2.1) *has a non-trivial solution branch* $(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n)$ *depending on a parameter* $\alpha \in R$ *so that* $y_r(0) = 0$, $\mu_r(0) = \lambda$, $1 \leqslant r \leqslant n$.

To conclude this paper we state the following completeness result. The proof parallels the arguments of [6] and is omitted.

THEOREM 3. *Under assumption* (1.3), *problem* (2.1) *has a sequence of eigenvalues* $\mu^m = (\mu_1^m, \ldots, \mu_n^m)$ *and corresponding eigenfunctions* $y_r^m(x_r; \mu^m)$ *such that* $(\prod_{r=1}^n y_r^m)_{m=1}^\infty$ *forms a basis for* $L^2([a, b], |A|)$. *That is, any* $f \in L^2([a, b], |A|)$ *may be expanded uniquely as*

$$f = \sum_{m=1}^{\infty} c_m \left(\prod_{r=1}^{n} y_r^m \right).$$

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